

# Discrete Newtonian Cosmology: Perturbations

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## Abstract

In a previous paper [1] we showed how a finite system of discrete particles interacting with each other via Newtonian gravitational attraction would lead to precisely the same dynamical equations for homothetic motion as in the case of the pressure-free Friedmann-Lemaître-Robertson-Walker cosmological models of General Relativity Theory, provided the distribution of particles obeys the central configuration equation. In this paper we show one can obtain perturbed such Newtonian solutions that give the same linearised structure growth equations as in the general relativity case. We also obtain the Dmitriev-Zeldovich equations for subsystems in this discrete gravitational model, and show how it leads to the conclusion that voids have an apparent negative mass.

## 1 Introduction

This is the second part of a treatment of Discrete Newtonian Cosmology based on a point particle model according to which, in contrast to the usual fluid models, the universe is conceived of as consisting of a large number  $N$  of gravitating point particles of mass  $m_a$  and positions  $\mathbf{x}_a(t)$  acted upon by Newtonian gravity and a possible cosmological term. In our first paper [1] we laid down the foundations and described how homothetic solutions  $\mathbf{x}_a = S(t)\mathbf{r}_a$  may be constructed which are the analogues of the Friedmann-Lemaître models of the continuum theory. The scale factor  $S(t)$  was shown to exactly satisfy the Raychaudhuri equation of gravitational attraction provided the co-moving positions  $\mathbf{r}_a$  constitute a *central configuration* (see (10)). In previous work [2] it has been shown that for  $N$  large and all masses  $m_a$  equal, there exist central configurations for which the point particles are distributed in an extremely homogeneous and isotropic fashion within a ball of finite radius. Thus one obtains the same results as in the fluid case, but without making the fluid assumption, which is somewhat dubious in this context [1]. After all most of the material content of the universe appears to be in the form of cold dark matter whose precise nature

is unknown except that it probably consists of a non-interacting gas of particles which interact solely by gravitational forces. In our Newtonian model we need only assume that the dominant material content of the universe consists of particles moving non-relativistically whose masses we need not specify and which interact solely by Newton's inverse square law of gravitation.

In this paper we investigate the behaviour of inhomogeneous discrete Newtonian cosmological models representing small deviations from that cosmological background. After reviewing the basic theory and the exact homothetic solutions in we shall, in Section 2, outline how perturbations around a general solution of Newton's equations of motion behave. We then apply this general theory to homothetic solutions, obtaining the discrete Newtonian analogue of perturbed relativistic cosmological models. This gives the same equations of motion as fluid-based Newtonian perturbation theory [3], which is also the same as in the pressure-free General Relativity case [4]. We go on in Section 4 to derive, following [5], what we call the *Dmitriev-Zeldovich equations*. This is a rather different approach to perturbation theory [6], in which we obtain equations governing the motion of Newtonian point particles in a background Friedmann-Lemaitre cosmology. This is a mean-field theory in which the point particles interact gravitationally with each other but have negligible effect on the background. The resulting equations are widely used in investigations of large-scale structure in cosmology [7, 8]. In Section 5, we relate this to the Swiss Cheese approximation used in General Relativity, and comment on the apparent negative mass of voids, in accordance with Newtonian work by Föppl and general relativity comments by Bondi.

In the remainder of this section we summarise the discrete Newtonian theory that was set out in [1], giving the general exact dynamic equations, plus the exact homothetic solution for the background cosmology.

## 1.1 Equations of motion

Consider an isolated set of gravitating particles, with no other interparticle forces. The gravitational force of the  $b$ -th particle on the  $a$ -th particle is

$$\mathbf{F}_{ab} = -\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) = -\mathbf{F}_{ba}, \quad (1)$$

where  $G$  is Newton's gravitational constant. The equation of motion for the  $a$ -th particle is

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = - \sum_{b \neq a} Gm_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} = \mathbf{F}_a \quad (2)$$

where  $\mathbf{F}_a$  is the total gravitational force acting on the  $a$ -th particle due to all the other particles in the system. It can be represented in terms of the gravitational

potential energy  $V_a$  of the particle  $a$  due to all the other particles, defined by

$$V_a(\mathbf{x}_a) := - \sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (3)$$

(this clearly depends on the position of the particle  $a$ ). The gravitational force on the  $a$ -th particle due to the system of particles is the gradient of this potential:

$$\frac{\partial V_a}{\partial \mathbf{x}_a} = - \sum_{b \neq a} \mathbf{F}_{ab} = -\mathbf{F}_a. \quad (4)$$

Because particle mass  $m_a$  is conserved, the equations are invariant under time reversal, time translations, spatial translations, and rotations. In accordance with Noether's theorem, there are conserved quantities associated with each of the three continuous symmetries. In particular, total energy  $\mathcal{E}$  of the set of particles is conserved:

$$\mathcal{E} = T + V = \mathcal{E}_0 \text{ (constant)}, \quad (5)$$

where the total kinetic energy  $T(\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dots, \dot{\mathbf{x}}_N)$  and the total potential energy  $V(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$  are defined by

$$T(\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dots, \dot{\mathbf{x}}_N) := \frac{1}{2} \sum_a m_a (\dot{\mathbf{x}}_a)^2 \quad (6)$$

$$V(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \frac{1}{2} \sum_a V_a = - \sum_{1 \leq a < b \leq N} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (7)$$

These are just single numbers for the entire set of particles: coarse-grained representations of its total internal state of motion and its total gravitational self-interaction. Thus neither is a function of position.

## 1.2 Homothetic ansatz

To obtain the background cosmological model, we assume self-similarity of the solution [1]. Then there is a homothetic factor  $S(t)$  such that

$$\mathbf{x}_a = S(t) \mathbf{r}_a, \quad d\mathbf{r}_a/dt = 0, \quad (8)$$

where  $\mathbf{r}_a$  are co-moving coordinates for the particle  $a$ . The total mass of matter  $M$  in a co-moving volume  $V$  is given by  $M_V := \sum_{a \in V} m_a$  which is conserved. The volume scales as  $V = S^3(t) V_0$  so the density scales as

$$\rho := \frac{M_V}{V} = \frac{M_V}{S^3(t) V_0} = \frac{\rho_0}{S^3(t)}, \quad (9)$$

where  $\rho_0 := \frac{M_V}{V_0}$ .

Define  $C(t) := S^2(t) \frac{d^2 S(t)}{dt^2}$  and substitute into the equation of motion (2); then consistency demands that  $C(t) = \text{const} =: -G\tilde{M}$ , where  $\tilde{M}$  is the effective gravitational mass of the system, and the equation separates into the *central configuration equation*

$$\tilde{M}m_a\mathbf{r}_a = \sum_{b \neq a} m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \quad (10)$$

which must hold for all values  $a$  ([2]; [9]:79-80), which is a consistency condition for (8) to give a solution, and the Raychaudhuri equation

$$-\frac{G\tilde{M}}{S^2(t)} = \frac{d^2 S(t)}{dt^2} \quad (11)$$

which gives the time evolution. Equation (10) determines the value of  $\tilde{M}$ , which is not the same as  $M_V$ . Defining the effective potential

$$\tilde{V}_{(-1)} := - \sum_{1 \leq a < b \leq N} \frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \quad (12)$$

of the total system of particles and its effective moment of inertia

$$\tilde{I}_0 := \frac{1}{2} \sum_a m_a (\mathbf{r}_a)^2 \quad (13)$$

in terms of the co moving  $\mathbf{r}_a$ , these are both constants. A key identity following from the central configuration equation is

$$2G\tilde{M}\tilde{I}_0 = -\tilde{V}_{(-1)}, \quad (14)$$

which can be used to determine  $\tilde{M}$ . In consequence of this identity, the energy conservation equation(5) is equivalent to the usual Friedmann equation

$$\frac{1}{2} \left[ \frac{\dot{S}(t)}{S(t)} \right]^2 = \frac{G\tilde{M}}{S^3(t)} + \frac{E}{S^2(t)} \quad (15)$$

for pressure-free matter, where  $E := \frac{\mathcal{E}_0}{2\tilde{I}_0}$  is a rescaled version of the total internal energy of the system, see (5). This is a first integral of the Raychaudhuri equation (11) .

## 2 Perturbations

In this section first we perturb the generic equations, and then apply that method to obtain a perturbed form of the homothetic solutions.

## 2.1 The general case

The general form of the equations of motion we consider is

$$m_a \ddot{\mathbf{x}}_a = - \frac{\partial V(\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_N)}{\partial \mathbf{x}_a}. \quad (16)$$

where  $V(\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_N)$  is the mutual gravitational potential energy of our  $N$  particles, given by (7). Actually this is a master equation that applies for any conservative kind of force; our specific application is where only gravitational forces act.

### 2.1.1 Potential form and Hessian

Now consider a background solution given by  $\bar{\mathbf{x}}_a$  and linear perturbation  $\delta \mathbf{y}_a$  about this solution, so that

$$\mathbf{x}_a = \bar{\mathbf{x}}_a + \delta \mathbf{y}_a, \quad |\bar{\mathbf{x}}_a| \gg |\delta \mathbf{y}_a|. \quad (17)$$

A simple use of Taylor's theorem, neglecting second order terms in  $\delta \mathbf{y}_a$  yields

$$\begin{aligned} m_a \left[ \frac{d^2(\bar{\mathbf{x}}_a)}{dt^2} + \frac{d^2(\delta \mathbf{y}_a)}{dt^2} \right] &= m_a \frac{d^2(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a)}{dt^2} \\ &= - \frac{\partial V(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a)}{\partial \mathbf{x}_a} \\ &= - \left[ \frac{\partial V(\bar{\mathbf{x}}_a)}{\partial \mathbf{x}_a} + \frac{\partial^2 V(\bar{\mathbf{x}}_a)}{\partial \mathbf{x}_a \partial \mathbf{x}_b} \cdot \partial \mathbf{x}_b \right]. \end{aligned} \quad (18)$$

Cancelling the background terms, the perturbation equation is

$$m_a \ddot{\delta \mathbf{y}}_a = - \sum_{b \neq a} \frac{\partial^2 V}{\partial \mathbf{x}_a \partial \mathbf{x}_b}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \dots \bar{\mathbf{x}}_N) \cdot \delta \mathbf{y}_b. \quad (19)$$

The symmetric linear operator acting on  $\mathbf{y}_a$  is in fact minus the Hessian  $\mathbf{E}_{ab}$  of  $V$ , considered as a function on the  $3N$ -dimensional configuration space evaluated on the background solution:

$$m_a \ddot{\delta \mathbf{y}}_a = \sum_{b \neq a} \mathbf{E}_{ab} \cdot \delta \mathbf{y}_b, \quad \mathbf{E}_{ab} := - \frac{\partial^2 V_a(\bar{\mathbf{x}}_c)}{\partial \mathbf{x}_a \partial \mathbf{x}_b}. \quad (20)$$

In general (19) or equivalently (20) is a linear ordinary differential equation for the perturbation  $\delta \mathbf{y}_a(t)$  whose coefficients depend on the background solution  $\bar{\mathbf{x}}_a(t)$ . These coefficients will in general therefore be time dependent. Equation (19) was obtained in the case of 4 particles undergoing a homothetic motion in [10], and an evaluation of the resulting Hessian carried out.

### 2.1.2 Force form

Using the expression  $\mathbf{F}_{ab} = -\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b)$  for the force between the particles at  $\mathbf{x}_a$  and  $\mathbf{x}_b$  and setting  $\mathbf{x}_{ba} := \mathbf{x}_b - \mathbf{x}_a$ ,  $\delta \mathbf{y}_{ba} = \delta \mathbf{y}_b - \delta \mathbf{y}_a$ ,  $x_{ba} := |\mathbf{x}_b - \mathbf{x}_a| = ((\mathbf{x}_b - \mathbf{x}_a) \cdot (\mathbf{x}_b - \mathbf{x}_a))^{1/2}$  gives

$$\begin{aligned} \mathbf{F}_{ab}(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) &= -\frac{Gm_a m_b}{|(\bar{\mathbf{x}}_b - \bar{\mathbf{x}}_a) + \delta \mathbf{y}_{ab}|^3}((\bar{\mathbf{x}}_a - \bar{\mathbf{x}}_b) + \delta \mathbf{y}_{ab}) \\ &= -\frac{Gm_a m_b}{|\bar{\mathbf{x}}_b - \bar{\mathbf{x}}_a|^3}(\bar{\mathbf{x}}_{ab} + \delta \mathbf{y}_{ab}) - \frac{\partial}{\partial \mathbf{x}_a} \left[ \frac{Gm_a m_b}{|\mathbf{x}_b - \mathbf{x}_a|^3} \right] \cdot \delta \mathbf{y}_{ab} \mathbf{x}_{ab} + \mathbf{O}(\delta \mathbf{y}_a)^2 \end{aligned}$$

to first order, where the partial derivative  $(\partial/\partial \mathbf{x}_a)$  is taken keeping all the other positions  $\mathbf{x}_b$  ( $b \neq a$ ) constant. For  $\mathbf{x}_a \neq \mathbf{x}_b$ ,

$$\frac{\partial}{\partial \mathbf{x}_a}(x_{ba}) = -\frac{1}{2}((\mathbf{x}_b - \mathbf{x}_a) \cdot (\mathbf{x}_b - \mathbf{x}_a))^{-1/2} 2(\mathbf{x}_b - \mathbf{x}_a) = -(x_{ba})^{-1} \mathbf{x}_{ba}$$

This gives

$$\begin{aligned} \delta \mathbf{F}_{ab} &= \mathbf{F}_{ab}(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) - \mathbf{F}_{ab}(\bar{\mathbf{x}}_a) \\ &= \frac{Gm_a m_b}{|\bar{\mathbf{x}}_{ab}|^3} \delta \mathbf{y}_{ba} - 3 \frac{Gm_a m_b}{|\bar{\mathbf{x}}_{ab}|^4} \left( \frac{\partial}{\partial \mathbf{x}_a}(x_{ba}) \cdot \delta \mathbf{y}_{ab} \right) \bar{\mathbf{x}}_{ba} \end{aligned} \quad (21)$$

$$= \frac{Gm_a m_b}{|\bar{\mathbf{x}}_{ab}|^5} \{ \delta \mathbf{y}_{ba} |\bar{\mathbf{x}}_{ab}|^2 - 3(\bar{\mathbf{x}}_{ba} \cdot \delta \mathbf{y}_{ba}) \bar{\mathbf{x}}_{ba} \} \quad (22)$$

and so

$$m_a(\delta \mathbf{y}_a) = \sum_{b \neq a} \frac{Gm_a m_b}{\bar{\mathbf{x}}_{ab}^5} \{ \delta \mathbf{y}_{ba} \bar{\mathbf{x}}_{ab}^2 - 3(\bar{\mathbf{x}}_{ba} \cdot \delta \mathbf{y}_{ba}) \bar{\mathbf{x}}_{ba} \} \quad (23)$$

This applies generically to perturbations about any background.

## 2.2 The cosmology case

We now apply the general formalism to the homothetically expanding background solution described in Section 1. Thus we have

$$\bar{\mathbf{x}}_a = S(t) \bar{\mathbf{r}}_a, \quad \bar{\mathbf{r}}_a = \text{const}, \quad \bar{\mathbf{r}}_{ab} := \bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b = \text{const}, \quad \bar{r}_{ab} := |\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b| = \text{const}. \quad (24)$$

Define co moving perturbation variables  $\mathbf{S}_a(t)$ ,  $\mathbf{S}_{ba}(t)$  by

$$\delta \mathbf{y}_a = S(t) \mathbf{S}_a(t), \quad \mathbf{S}_{ba}(t) := \mathbf{S}_b - \mathbf{S}_a. \quad (25)$$

Then eqn (23) becomes

$$\begin{aligned} m_a \frac{d^2}{dt^2}(S(t) \mathbf{S}_a) &= \sum_{b \neq a} \frac{Gm_a m_b}{S^5(t) |\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^5} S^3(t) \{ (\mathbf{S}_b - \mathbf{S}_a) |\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^2 \\ &\quad - 3(\bar{\mathbf{r}}_b - \bar{\mathbf{r}}_a) \cdot \mathbf{S}_{ba} (\bar{\mathbf{r}}_b - \bar{\mathbf{r}}_a) \}, \end{aligned} \quad (26)$$

giving the cosmological perturbation equation

$$S^2 m_a \frac{d^2}{dt^2} (S \mathbf{S}_a) = \sum_{b \neq a} \frac{G m_a m_b}{|\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^5} \{ \bar{r}_{ba}^2 \mathbf{S}_{ba} - 3(\bar{\mathbf{r}}_{ba} \cdot \mathbf{S}_{ba}) \bar{\mathbf{r}}_{ba} \} \quad (27)$$

As in the general case discussed earlier (27) is a second order ordinary differential equation for the perturbation  $\mathbf{S}_a(t)$  whose coefficients depend upon the background scale factor  $S(t)$  and the background time independent central configuration  $\bar{\mathbf{r}}_a$  whose homothetic expansion we are perturbing about. Since we are not changing the masses  $m_a$  in the central configuration equation its solutions, which are critical points of a fixed function on configuration space, will generically be isolated, and so in fact there are no static small perturbations of the central configuration equation to consider.

### 2.2.1 Asymptotic solution

Multiply by  $(1/m_a S^2)$ , the growth of perturbations is given by

$$\frac{d^2}{dt^2} (S \mathbf{S}_a) = \frac{1}{S^2} \sum_{b \neq c} \frac{G m_b}{\bar{r}_{ab}^5} \{ \bar{r}_{ba}^2 \mathbf{S}_{ba} - 3(\bar{\mathbf{r}}_{ba} \cdot \mathbf{S}_{ba}) \bar{\mathbf{r}}_{ba} \} \quad (28)$$

The right hand side goes to zero as  $S \rightarrow \infty$ . Thus at late times

$$S \mathbf{S}_a = \mathbf{w}_a t + \mathbf{q}_a \quad (29)$$

where  $\mathbf{w}_a, \mathbf{q}_a$  are constant vectors, and so, because  $S \propto t^{2/3}$ ,

$$\mathbf{S}_a = \tilde{\mathbf{w}}_a t^{1/3} + \frac{\tilde{\mathbf{q}}_a}{t^{2/3}}. \quad (30)$$

The first term grows only algebraically, while the second term decays, so eventually  $\mathbf{S}_a \propto t^{1/3}$ . The magnitude of the change is

$$\mathbf{S}^2 = \mathbf{S}_a \mathbf{S}_a = \left( \tilde{\mathbf{w}}_a t^{1/3} + \frac{\tilde{\mathbf{q}}_a}{t^{2/3}} \right) \left( \tilde{\mathbf{w}}_a t^{1/3} + \frac{\tilde{\mathbf{q}}_a}{t^{2/3}} \right). \quad (31)$$

so at late times  $\mathbf{S}^2 = \tilde{\mathbf{w}}^2 t^{2/3}$ .

## 2.3 The density perturbation equation

The mass of matter  $M$  in a co moving volume  $V$  is given by  $M_V := \sum_{a \in V} m_a$ ,

which is conserved when the system is perturbed (particle mass is unchanged). But then  $V = S^3(t)(V_0 + \delta V)$  where  $\delta V$  is found by choosing three vectors  $\mathbf{x}_{ab}^i, \mathbf{x}_{ac}^j, \mathbf{x}_{ad}^k$  linking particle  $a$  to particles  $b, c, d$ . The volume defined by these particles is

$$\begin{aligned}
V_{abcd} &= \varepsilon_{ijk} \mathbf{x}_{ab}^i \mathbf{x}_{ac}^j \mathbf{x}_{ad}^k = \varepsilon_{ijk} (\bar{\mathbf{x}}_{ab}^i + \delta \mathbf{y}_{ab}^i) (\bar{\mathbf{x}}_{ac}^j + \delta \mathbf{y}_{ac}^j) (\bar{\mathbf{x}}_{ad}^k + \delta \mathbf{y}_{ad}^k) \\
&= \bar{V}_{abcd} + \varepsilon_{ijk} (\bar{\mathbf{x}}_{ac}^j \bar{\mathbf{x}}_{ad}^k \delta \mathbf{y}_{ab}^i + \bar{\mathbf{x}}_{ad}^k \bar{\mathbf{x}}_{ab}^i \delta \mathbf{y}_{ac}^j + \bar{\mathbf{x}}_{ab}^i \bar{\mathbf{x}}_{ac}^j \delta \mathbf{y}_{ad}^k) + o(\delta^2)
\end{aligned}$$

In the cosmological case this is

$$V_{abcd} = \bar{V}_{abcd} + S^3(t) \varepsilon_{ijk} (\bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ad}^k S_{ab}^i + \bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ab}^i S_{ac}^j + \bar{\mathbf{r}}_{ab}^i \bar{\mathbf{r}}_{ac}^j S_{ad}^k)$$

At late times they obey (29) so the volume  $\delta V$  behaves as

$$\begin{aligned}
\delta V &= S^3(t) \varepsilon_{ijk} (\bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ad}^k S_{ab}^i + \bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ab}^i S_{ac}^j + \bar{\mathbf{r}}_{ab}^i \bar{\mathbf{r}}_{ac}^j S_{ad}^k), \\
\mathbf{S}_{ab}^i &:= (\mathbf{w}_a^i t + \mathbf{q}_a^i) - (\mathbf{w}_b^i t + \mathbf{q}_b^i) = (\mathbf{w}_a^i - \mathbf{w}_b^i) t + (\mathbf{q}_a^i - \mathbf{q}_b^i)
\end{aligned}$$

Thus their density changes as

$$\rho := \frac{M}{(V + \delta V)} \approx \frac{M}{S^3(t)} (1 - \varepsilon_{ijk} (\bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ad}^k S_{ab}^i + \bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ab}^i S_{ac}^j + \bar{\mathbf{r}}_{ab}^i \bar{\mathbf{r}}_{ac}^j S_{ad}^k)) = \rho + \delta \rho,$$

So finally density perturbations overall for large  $t$  are given by

$$\begin{aligned}
\frac{\delta \rho}{\rho} &= -\frac{1}{n} \sum_{a,b,c,d} \varepsilon_{ijk} (\bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ad}^k (\mathbf{w}_a^i - \mathbf{w}_b^i) t \\
&\quad + \bar{\mathbf{r}}_{ac}^j \bar{\mathbf{r}}_{ab}^i (\mathbf{w}_a^i - \mathbf{w}_c^i) t + \bar{\mathbf{r}}_{ab}^i \bar{\mathbf{r}}_{ac}^j (\mathbf{w}_a^i - \mathbf{w}_d^i) t) \\
&\approx W t \propto S^{3/2}
\end{aligned} \tag{32}$$

$W$  depends on initial conditions. If  $W > 0$  we have the growth of an over-density, if  $W < 0$  the growth of an under density or void.

### 3 Cosmological constant

The universe appears today to be dominated by a cosmological constant. Adding in a Newtonian cosmological constant to the force law, we get

$$m_a \frac{d^2 \mathbf{x}_a}{dt^2} = - \sum_{b \neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \frac{\Lambda m_a \mathbf{x}_a}{3}. \tag{33}$$

#### 3.0.1 Perturbations with cosmological constant

Now consider a background solution given by  $\bar{\mathbf{x}}_a$  and linear perturbation  $\delta \mathbf{y}_a$  about this solution, so that as before,  $\mathbf{x}_a = \bar{\mathbf{x}}_a + \delta \mathbf{y}_a$ ,  $|\bar{\mathbf{x}}_a| \gg |\delta \mathbf{y}_a|$ . Again, a simple use of Taylor's theorem, neglecting second terms in  $\delta \mathbf{y}_a$  yields



$$m_a \frac{d^2(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a)}{dt^2} = m_a \left[ \frac{d^2(\bar{\mathbf{x}}_a)}{dt^2} + \frac{d^2(\delta \mathbf{y}_a)}{dt^2} \right] + \frac{\Lambda m_a (\bar{\mathbf{x}}_a + \delta \mathbf{y}_a)}{3} \quad (34)$$

$$= - \frac{\partial V_a(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a)}{\partial \mathbf{x}_a} = - \left[ \frac{\partial V_a(\bar{\mathbf{x}}_a)}{\partial \mathbf{x}_a} + \frac{\partial^2 V_a(\bar{\mathbf{x}}_a)}{\partial \mathbf{x}_a \partial \mathbf{x}_b} \partial \mathbf{x}_b \right]. \quad (35)$$

where the potential  $V_{a\Lambda}$  and its derivatives are

$$V_{a\Lambda} := \sum_{a,i} \frac{\Lambda}{6} m_a (x_a^i)^2 := \sum_{a,i} \frac{\Lambda}{6} m_a (\bar{\mathbf{x}}_a + \delta \mathbf{y}_a^i)^2, \quad (36)$$

$$\frac{\partial V_{a\Lambda}}{\partial x_{ai}} := \sum_{a,i} \frac{\Lambda}{3} m_a (\delta \mathbf{y}_a^i), \quad (37)$$

$$\frac{\partial^2 V_{a\Lambda}}{\partial x_{ai} \partial x_{bj}} = \frac{\partial}{\partial x_{bj}} \sum_{a,i} \frac{\Lambda}{3} m_a (\delta \mathbf{y}_a^i) = \frac{\Lambda}{3} m_a \delta_{ab} \delta_{ij}. \quad (38)$$

Cancelling the background terms, the perturbation equation is

$$m_a \ddot{\delta \mathbf{y}}_a = - \sum_{b \neq a} \frac{\partial^2 V_a}{\partial \mathbf{x}_a \partial \mathbf{x}_b} (\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \dots \bar{\mathbf{x}}_N) \delta \mathbf{y}_b + m_a \frac{\Lambda}{3} \delta \mathbf{y}_a.$$

The symmetric linear operator acting on  $\mathbf{y}_a$  is minus the Hessian of  $V = V_a^{grav} + V_{a\Lambda}$ , considered as a function on the 3N-dimensional configuration space evaluated on the background solution.

### 3.0.2 Force form

Using the expression  $\mathbf{F}_{ab} = -\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3} (\mathbf{x}_a - \mathbf{x}_b) + \frac{\Lambda m_a \mathbf{x}_a}{3}$  for the force between the particles at  $\mathbf{x}_a$  and  $\mathbf{x}_b$  and proceeding as before gives

$$\begin{aligned} \mathbf{F}_{ab}(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) &= - \frac{Gm_a m_b}{|(\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) - (\bar{\mathbf{x}}_b + \delta \mathbf{y}_b)|^3} ((\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) - (\bar{\mathbf{x}}_b + \delta \mathbf{y}_b)) \\ &+ \frac{\Lambda m_a}{3} (\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) \\ &= - \frac{Gm_a m_b}{|\bar{\mathbf{x}}_b - \bar{\mathbf{x}}_a|^3} (\bar{\mathbf{x}}_{ab} + \delta \mathbf{y}_{ab}) - \frac{\partial}{\partial \mathbf{x}_a} \left[ \frac{Gm_a m_b}{|\mathbf{x}_b - \mathbf{x}_a|^3} \right] (\bar{\mathbf{x}}_{ab}) (\delta \mathbf{y}_a) \\ &+ \frac{\Lambda m_a}{3} (\bar{\mathbf{x}}_a + \delta \mathbf{y}_a) + \mathbf{O}(\delta \mathbf{y}_a)^2 \end{aligned} \quad (39)$$

to first order, where the partial derivative  $(\partial/\partial \mathbf{x}_a)$  is taken keeping all the other positions  $\mathbf{x}_b$  ( $b \neq a$ ) constant. This gives

$$\delta \mathbf{F}_{ab} = \frac{Gm_a m_b}{|\bar{\mathbf{x}}_{ab}|^5} \{ \delta \mathbf{y}_{ba} |\bar{\mathbf{x}}_{ab}|^2 - 3(\bar{\mathbf{x}}_{ba})(\bar{\mathbf{x}}_{ba}) \cdot \delta \mathbf{y}_{ba} \} + \frac{\Lambda m_a}{3} \delta \mathbf{y}_a$$

and so

$$m_a(\delta \mathbf{y}_a)^\cdot = \sum_{b \neq a} \frac{Gm_a m_b}{|\bar{\mathbf{x}}_a - \bar{\mathbf{x}}_b|^5} \{ \delta \mathbf{y}_{ba} |\bar{\mathbf{x}}_a - \bar{\mathbf{x}}_b|^2 - 3(\bar{\mathbf{x}}_b - \bar{\mathbf{x}}_a)((\bar{\mathbf{x}}_b - \bar{\mathbf{x}}_a) \cdot \delta \mathbf{y}_{ba}) \} + \frac{\Lambda m_a}{3} \delta \mathbf{y}_a \quad (40)$$

### 3.0.3 Background cosmology with cosmological constant

As before, put in a homothetic factor and separate variables: using (8), (33) becomes

$$m_a \mathbf{r}_a \frac{d^2 S(t)}{dt^2} = - \sum_{b \neq a} Gm_a m_b \frac{S(t)(\mathbf{r}_a - \mathbf{r}_b)}{S^3(t)|\mathbf{r}_a - \mathbf{r}_b|^3} + \frac{\Lambda S(t)m_a \mathbf{r}_a}{3}. \quad (41)$$

The argument goes through as before. This gives the result

$$m_a \mathbf{r}_a S^2(t) \frac{d^2 S(t)}{dt^2} = -G\tilde{M}m_a \mathbf{r}_a + \frac{\Lambda S^3(t)m_a \mathbf{r}_a}{3} \quad (42)$$

with  $\tilde{M}$  defined exactly as before by (10). This implies the Raychaudhuri equation with cosmological constant:

$$\frac{1}{S(t)} \frac{d^2 S(t)}{dt^2} = -\frac{G\tilde{M}}{S^3(t)} + \frac{\Lambda}{3} \quad (43)$$

where matter causes deceleration and  $\Lambda$  an acceleration. To integrate when  $dS/dt \neq 0$ , multiply by  $S(t)dS/dt$  to get the Friedmann equation

$$\frac{1}{2} \left[ \frac{\dot{S}(t)}{S(t)} \right]^2 = \frac{G\tilde{M}}{S^3(t)} + \frac{E}{S^2(t)} + \frac{\Lambda}{6} \quad (44)$$

where  $E$  is a constant of integration.

### 3.0.4 Perturbed cosmology with cosmological constant

We again apply this general formalism to the homothetically expanding background solution  $\bar{\mathbf{x}}_a = S(t)\bar{\mathbf{r}}_a$ ,  $\bar{\mathbf{r}}_a = \text{const}$  and define  $\delta \mathbf{y}_a = S(t)\mathbf{S}_a(t)$ . Then

$$m_a \frac{d^2}{dt^2} (S(t)\mathbf{S}_a) = \sum_{b \neq c} \frac{Gm_a m_b}{S^5(t)|\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^5} S^3(t) \{ (\mathbf{S}_b - \mathbf{S}_a) |\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^2 - 3(\bar{\mathbf{r}}_b - \bar{\mathbf{r}}_a) \cdot \mathbf{S}_{ba} (\bar{\mathbf{r}}_b - \bar{\mathbf{r}}_a) \} + \frac{\Lambda m_a}{3} S(t)\mathbf{S}_a(t) \quad (45)$$

giving the cosmological perturbation equation

$$S^2 m_a \frac{d^2}{dt^2} (S\mathbf{S}_a) = \sum_{b \neq c} \frac{Gm_a m_b}{|\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b|^5} \{ \bar{r}_{ba}^2 \mathbf{S}_{ba} - 3(\bar{\mathbf{r}}_{ba} \cdot \mathbf{S}_{ba}) \bar{\mathbf{r}}_{ba} \} + \frac{\Lambda m_a}{3} S^3(t)\mathbf{S}_a(t) \quad (46)$$

for perturbations with  $\Lambda \neq 0$ .

### 3.0.5 Asymptotic solution

Multiply by  $(1/m_a S^2)$ , the growth of perturbations is given by

$$\frac{d^2}{dt^2}(S \mathbf{S}_a) = \frac{1}{S^2} \sum_{b \neq c} \frac{G m_b}{\bar{r}_{ab}^5} \{ \bar{r}_{ba}^2 \mathbf{S}_{ba} - 3(\bar{\mathbf{r}}_{ba} \cdot \mathbf{S}_{ba}) \bar{\mathbf{r}}_{ba} \} + \frac{\Lambda}{3} S(t) \mathbf{S}_a(t) \quad (47)$$

The first term on the right hand side goes to zero as  $S \rightarrow \infty$ . Thus at late times

$$\frac{d^2}{dt^2}(S \mathbf{S}_a) = \frac{\Lambda}{3} (S \mathbf{S}_a). \quad (48)$$

Assuming  $\Lambda > 0$ , this implies

$$\mathbf{S}_a = \frac{\mathbf{S}_0 \exp \sqrt{\frac{\Lambda}{3}}(t - t_0)}{S(t)}.$$

where  $\mathbf{S}_0$  is a constant vector. This means the density perturbation is rapidly decreasing, as the exponential wins at late times:

$$\frac{d\mathbf{S}_a}{dt} = \frac{\mathbf{S}_0}{S(t)} \left[ \sqrt{\frac{\Lambda}{3}} - \frac{1}{S(t)} \frac{dS}{dt} \right] \exp \left( \sqrt{\frac{\Lambda}{3}}(t - t_0) \right)$$

which changes sign when  $\frac{1}{S(t)} \frac{dS}{dt} = \sqrt{\frac{\Lambda}{3}}$ . This is when the vacuum energy wins over the gravitational attraction, and structure formation ceases.

## 4 The Dmitriev-Zel'dovich equations

We turn now to a different approach to deriving perturbation equations, based in work of Dmitriev and Zeldovich, that is useful in n-body simulations [11].

One can group particles together to get identified subgroups, and coarse grain to get equations for each subgroup. Then one can assume one subgroup - say a system of galaxies - has little influence on the rest of the universe, which is much larger; so this system moves in the averaged field of the background universe, which is unaffected by its presence. In the case of just one subgroup, this gives the Dmitriev-Zeldovich equations from Newton's equations of motion, which are valid even when the situation is non-linear. This is the subject of sections (4.1) and (4.2).

The Dmitriev-Zel'dovich equations contain the scale factor  $S(t)$  and are thus time dependent. They nevertheless admit a Lagrangian description (discussed in section 4.3) and as a consequence satisfy the conservation of momentum and angular momentum by virtue of the translation and rotation invariance of the Lagrangian, although the expressions for the momentum and angular momentum in terms of position and velocities are time dependent because they

contain the scale factor  $S(t)$ . Because of the time dependence, energy is no longer conserved, and as we discuss in section (4.4) the usual Virial Theorem takes a modified form which is widely used in large scale structure studies.

The background Newtonian universe we are considering is not invariant under Galilean boosts and thus may be said to exhibit the spontaneous breakdown of Galilean invariance just as its relativistic version, the Friedmann-Lemaitre-Robertson-Walker metric exhibits the spontaneous breakdown of Lorentz invariance. Nevertheless, there remains a remnant of Galilean invariance in the Dmitriev-Zel'dovich equations, which exhibit a form of the relativity principle which has some relevance for discussions of whether space is relative or absolute. This is discussed in section (4.5).

In section (4.6) we discuss the two-body problem according to the Dmitriev-Zel'dovich equations and show how, in the adiabatic approximation, the orbits of planets around the sun or stars around the galaxy participate in the general expansion of the universe.

## 4.1 Coarse Graining

We start with the exact equations of motion for a large but finite number of particles:

$$m_a \ddot{\mathbf{x}}_a = \sum_{b \neq a} \frac{G m_a m_b (\mathbf{x}_b - \mathbf{x}_a)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \quad (49)$$

and assume that the particles fall into two classes, with  $a = i, j, k, \dots$  and  $a = I, J, K, \dots$ . The second set form a cosmological background and we make the approximation that their motion is unaffected by the first class of particles, galaxies, whose motion is however affected both by the background particles and their mutual attractions. Thus the equations of motion (49) split into two sets

$$m_I \ddot{\mathbf{x}}_I = \sum_{J \neq I} \frac{G m_I m_J (\mathbf{x}_J - \mathbf{x}_I)}{|\mathbf{x}_J - \mathbf{x}_I|^3} \quad (50)$$

for the background model and

$$m_i \ddot{\mathbf{x}}_i = \sum_{j \neq i} \frac{G m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{|\mathbf{x}_j - \mathbf{x}_i|^3} + \sum_J \frac{G m_i m_J (\mathbf{x}_J - \mathbf{x}_i)}{|\mathbf{x}_J - \mathbf{x}_i|^3} \quad (51)$$

for the subgroup. We now assume that the background particles move isometrically:

$$\mathbf{x}_I = S(t) \mathbf{r}_I. \quad (52)$$

Then by the above argument, they must form a central configuration and  $S(t)$  obeys the Friedmann equation (15). The deviation of the first set of particles from this mean Hubble flow is given by

$$m_i \ddot{\mathbf{x}}_i = \sum_{j \neq i} \frac{G m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{|\mathbf{x}_i - \mathbf{x}_j|^3} + \sum_J \frac{G m_i m_J S(t) (\mathbf{r}_J - \mathbf{r}_i)}{|S(t) (\mathbf{r}_J - \mathbf{r}_i)|^3} \quad (53)$$

We replace the absolute positions of the galaxies by the conformally scaled positions  $\mathbf{x}_i = S(t)\mathbf{r}_i(t)$  and obtain

$$\begin{aligned} m_i \left( S(t)\ddot{\mathbf{r}}_i + 2\dot{S}(t)\dot{\mathbf{r}}_i + \ddot{S}(t)\mathbf{r}_i \right) &= \frac{1}{S^2(t)} \sum_{j \neq i} \frac{Gm_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \\ &+ \frac{1}{S^2(t)} \sum_J \frac{Gm_i m_J (\mathbf{r}_J - \mathbf{r}_i)}{|\mathbf{r}_J - \mathbf{r}_i|^3}. \end{aligned} \quad (54)$$

The second term on the right hand side of (54) is the force  $F_i$  exerted on the  $i$ th galaxies by the background particles. The numerical work in [2] provided very good evidence that for a large number of background particles, the central configuration is to a very good approximation statistically spherically symmetric and homogeneous. It follows that the force exerted by the background is radial

$$\frac{1}{S^2(t)} \sum_J \frac{Gm_i m_J (\mathbf{r}_J - \mathbf{r}_i)}{|\mathbf{r}_J - \mathbf{r}_i|^3} = -G\tilde{M}m_i \mathbf{r}_i, \quad (55)$$

where by (11),

$$S^2 \ddot{S} = -G\tilde{M}. \quad (56)$$

Then the force term  $F_i := \frac{1}{S^2(t)} \sum_J \frac{Gm_i m_J (\mathbf{r}_J - \mathbf{r}_i)}{|\mathbf{r}_J - \mathbf{r}_i|^3}$  on the right hand side of (54) cancels the third term on the left hand side. We are left with

$$m_i \left( S(t)\ddot{\mathbf{r}}_i + 2\dot{S}(t)\dot{\mathbf{r}}_i \right) = \frac{1}{S^2(t)} \sum_{j \neq i} \frac{Gm_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad (57)$$

that is

$$\frac{d(S^2(t)\dot{\mathbf{r}}_i)}{dt} = \frac{1}{S(t)} \sum_{j \neq i} \frac{Gm_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad (58)$$

which are the Dmitriev-Zel'dovich equations [6].

Writing this in terms of inertial coordinates  $\mathbf{x}_i = S(t)\mathbf{r}_i$  rather than co-moving coordinates the Dmitriev-Zel'dovich equation takes the equivalent form

$$\ddot{\mathbf{x}}_i = \frac{\ddot{S}}{S} \mathbf{x}_i + \sum_{j \neq i} \frac{Gm_j (\mathbf{x}_j - \mathbf{x}_i)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \quad (59)$$

which appears in the work of [12, 13, 14].

The equations of motion (58,59) contain the time dependent scale factor  $S(t)$  and its first (58) or second (59) time derivative. nevertheless it is still possible to apply the standard techniques of Lagrangian and Hamiltonian mechanics as we shall show in the next subsection.

## 4.2 Lagrangian version

Peebles [15] has shown that the Dmitriev-Zel'dovich equation (58) may be derived from the (time-dependent) Lagrangian

$$L = \frac{1}{2}S^2 \sum_{1 \leq i \leq N} m_i \dot{\mathbf{r}}_i^2 + \frac{1}{S} \sum_{1 \leq i < j \leq N} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} = T - V, \quad (60)$$

with

$$T = \frac{1}{2} \sum_{1 \leq i \leq N} S^2 m_i \dot{\mathbf{r}}_i^2, \quad (61)$$

$$V = -\frac{1}{S} \sum_{1 \leq i < j \leq N} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (62)$$

The Lagrangian (60) differs from the Lagrangian

$$\tilde{L} = \frac{1}{2} \sum_{1 \leq i \leq N} (m_i \dot{\mathbf{x}}_i^2 + \frac{\ddot{S}}{S} m_i \mathbf{x}_i^2) + \sum_{1 \leq i < j \leq N} \frac{Gm_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}, \quad (63)$$

where  $\mathbf{r}_i = \frac{\mathbf{x}_i}{S(t)}$ , by a total time derivative. Therefore it should give rise to the same equations of motion. This is easily checked since the Euler Lagrange equations of  $\tilde{L}$  are in fact (59).

By virtue of translation and rotational invariance of  $L$ , the equations of motion conserve total momentum

$$\mathbf{P} = \sum_{1 \leq i \leq N} \mathbf{p}_i = S^2 \sum_{1 \leq i \leq N} m_i \dot{\mathbf{r}}_i \Rightarrow d\mathbf{P}/dt = 0, \quad (64)$$

and total angular momentum

$$\mathbf{L} = \sum_{1 \leq i \leq N} \mathbf{r}_i \times \mathbf{p}_i \Rightarrow d\mathbf{L}/dt = 0 \quad (65)$$

with

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \frac{\partial T}{\partial \dot{\mathbf{r}}_i} = S^2 m_i \dot{\mathbf{r}}_i. \quad (66)$$

## 4.3 Energy theorem and virial theorem

Because of the time dependence of the Lagrangian, the energy or Hamiltonian  $H$  is not conserved. For a general Lagrangian system we have

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad (67)$$

where

$$H = \sum_{1 \leq i \leq N} \mathbf{p}_i \cdot \dot{\mathbf{r}}_i - L \quad (68)$$

In our case

$$H = T + V \quad (69)$$

$$= \frac{1}{S^2(t)} \sum_{1 \leq j \leq N} \frac{\mathbf{p}_i^2}{2m_i} - \frac{1}{S} \sum_{1 \leq i < j \leq N} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (70)$$

and we have the so-called *Cosmic Energy Theorem* [16, 17, 6]

$$\frac{dH}{dt} = \frac{\dot{S}}{a} (2T - V) \quad (71)$$

Note that if  $V$  can be neglected, or for freely moving non-relativistic particles, the energy is pure kinetic and redshifts as  $\frac{1}{S^2}$ .

One may easily extend this result to include a cosmological term or possible dark energy effects cf. [18] or [19].

To obtain the so-called *Cosmic Virial Theorem* [20] we recall that for a general Lagrangian system

$$\frac{d}{dt} \sum_{1 \leq i \leq N} \mathbf{p}_i \cdot \mathbf{r}_i = \sum_{1 \leq i \leq N} \dot{\mathbf{p}}_i \cdot \mathbf{r}_i + \sum_{1 \leq i \leq N} \dot{\mathbf{p}}_i \mathbf{r}_i \quad (72)$$

$$= H + L + \sum_{1 \leq i \leq N} \dot{\mathbf{r}}_i \cdot \frac{\partial L}{\partial \mathbf{r}_i} \quad (73)$$

In our case we get

$$\frac{d}{dt} \left( S^2 \frac{dI}{dt} \right) = 2T + V \quad (74)$$

where

$$I = \frac{1}{2} \sum_{1 \leq i \leq N} m_i \mathbf{r}_i^2. \quad (75)$$

If we time average and assume that the average of the rhs is zero we get

$$\langle 2T + V \rangle = 0 \Rightarrow S^2 \langle \frac{dI}{dt} \rangle = \text{const} \quad (76)$$

which is the standard result (see [1]). This will be true when the local system has decoupled from the cosmic expansion; otherwise we get

$$(S^2 \langle dI/dt \rangle|_{t=t_1} - S^2 \langle dI/dt \rangle|_{t=t_2}) = \int_{t_1}^{t_2} (\langle 2T + V \rangle) dt \quad (77)$$

which will be non-zero for systems coupled to the cosmic expansion. Conditions for the Virial Theorem condition on the left of (76) to hold are given in [21]. In essence the result holds because the asymptotic average of the derivative of a bounded function is necessarily zero, thus it will hold for any bound system of self-gravitating particles.<sup>1</sup>

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<sup>1</sup>See <http://www.mathpages.com/home/kmath572/kmath572.htm> for more details.

## 4.4 Galilean Invariance

The equations of motion are invariant under the generalised Galilean transformations

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{a}(t) \quad (78)$$

where

$$\frac{d(S^2(t)\dot{\mathbf{a}})}{dt} = 0. \quad (79)$$

The Lagrangian itself is not invariant under (78) but changes by a time derivative. We also have that the centre of mass  $\mathbf{R}$ , defined by

$$\mathbf{R} = \frac{1}{M} \sum_{1 \leq i \leq N} m_i \mathbf{r}_i, \quad (80)$$

moves as

$$\frac{d(S^2(t)\dot{\mathbf{R}})}{dt} = 0. \quad (81)$$

and that by means of a generalised Galilean transformation of the form (78) we may pass to Barycentric coordinates for which  $\mathbf{R} = 0$ .

It is well known that while Leibnitz adhered to a relational theory of space, i.e. that absolute positions are unobservable, Newton appears to his critics at least to have favoured the idea that space is absolute or perhaps more accurately, that something (God?) determines an absolute standard of rest. In the late nineteenth century, by which time the proper definition and consequent arbitrariness of an inertial frame was finally understood [23, 22, 24, 25, 26, 27], there were also suggestions [22] that despite the fact that fundamental laws of dynamics were Galilei invariant, a privileged inertial frame, sometimes called the ‘Body Alpha’ [28] might be identified with the rest frame of all of the particles in the universe, assumed finite and that may always refer the fundamental equations of dynamics to that frame. As noted above, from a modern perspective, according to which our background universe spontaneously breaks Galilei invariance with a cosmological rest frame defined by the cosmic background radiation, the puzzle is why the motion of bodies within it should exhibit an albeit modified form of Galilei invariance. The answer we see at the Newtonian level is that it is inherited from the underlying Galilei invariance of the equations (2) which we started with.

From a more practical viewpoint it is worth perhaps worth remarking that the 2nd realisation of the International Celestial Reference System (ICRF2) uses 3414 extragalactic radio sources observed by Very Long Baseline Interferometry (VLBI) each of whose motion is presumably governed by the Dmitriev-Zeldovich equations.



## 4.5 Two-Body Problem for the Dmitriev-Zeldovich equations

The Dmitriev-Zeldovich equations for two bodies may be obtained from the Lagrangian

$$L = \frac{1}{2}S^2(t)(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2) + \frac{Gm_1m_2}{S(t)|\mathbf{r}_1 - \mathbf{r}_2|} \quad (82)$$

This may be re-arranged to give

$$L = \frac{1}{2}S^2(t)M\left(\frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2}\right)^2 + \frac{1}{2}S^2(t)\mu\dot{\mathbf{r}}^2 + \frac{1}{S(t)}\frac{GM\mu}{|\mathbf{r}|} \quad (83)$$

where  $M = m_1 + m_2$ ,  $\mu = \frac{m_1m_2}{m_1+m_2}$ ,  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ . The first term give the motion of the centre of mass  $\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$  and the second and third terms the relative motion. If  $S(t) = \text{constant}$  this is the standard Kepler problem. If  $S(t)$  varies with time, except for the special Vinti-Lynden-Bell case in which the solution may be expressed in terms of the solution of the time independent case (see [33, 34, 35]), we must resort to an approximation. However relative angular momentum  $\mathbf{L}$  is conserved and by rotational invariance we may still reduce the problem to one in the equatorial plane orthogonal to  $\mathbf{L}$ .

### 4.5.1 Adiabatic Invariants

The standard approach to problems of this kind is to find the adiabatic invariants of the time-independent motion and then, for slowly  $S(t)$  they should be constant. The relevant adiabatic invariants are  $\frac{1}{2\pi} \oint p_r dr$  and  $\frac{1}{2\pi} \oint p_\phi d\phi$ .

The motion with  $S(t)$  constant is an ellipse with semi-major axis  $a$  and eccentricity  $\epsilon$  and semi-major axis  $b = a\sqrt{1-\epsilon^2}$  and semi-latus rectum  $l = a(1-\epsilon^2)$ .

$$\frac{1}{r} = \frac{1 + \epsilon \cos \phi}{a(1 - \epsilon^2)} \mathbf{r} \quad (84)$$

and so we want to know how  $\epsilon$  and  $a$  vary with time in the adiabatic approximation. We have

$$\frac{1}{2\pi} \oint p_r dr = p_\phi \left( \frac{1}{\sqrt{1-\epsilon^2}} - 1 \right). \quad (85)$$

An illuminating derivation of (85) is to be found in [37, 38] as follows. Let  $p$  and  $p'$  be the perpendicular distances from the foci  $F$  and  $F'$  of an ellipse to the tangent at the point  $P$  whose focal distances are  $r$  and  $r'$ . Since the two focal radii are equally inclined to the tangent we have

$$\frac{r}{p} = \frac{r'}{p'}. \quad (86)$$

Now the pedal equations of the ellipse, respect to the foci are

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a}, \quad \frac{l}{p'^2} = \frac{2}{r'} - \frac{1}{a}, \quad (87)$$

where  $l = a(1 - \epsilon^2)$  is the semi-latus rectum and  $a$  the semi-major axis, and  $\epsilon$  its eccentricity. Thus

$$\frac{lr'}{pp'} = 2 - \frac{r}{a}, \quad \frac{lr}{pp'} = 2 - \frac{r'}{a}, \quad (88)$$

Addition yields

$$\frac{l(r + r')}{pp'} = 2 - \frac{r + r'}{a}. \quad (89)$$

But since  $r + r' = 2a$ , it follows that

$$pp' = b^2, \quad (90)$$

where  $b = a\sqrt{1 - \epsilon^2}$  is the semi-minor axis.

Now consider a particle moving in an elliptic orbit about the focus  $F$ . It is well known that Kepler's third law states that area  $A$  swept out by the radius vector from the focus  $F$  is proportional to the time:

$$\int dA = \frac{1}{2} \int p ds = \frac{1}{2} \int p v dt = \frac{1}{2} h \int dt, \quad (91)$$

where  $h = pv = \frac{1}{m}p_\phi$  is the angular momentum per unit mass.

Less well known is the fact [37, 38] that the area  $A'$  swept out by the radius vector from the focus  $F'$  is proportional to the action:

$$\int dA' = \frac{1}{2} \int p' ds = \frac{1}{2} \int \frac{b^2}{p} ds = \frac{1}{2} \frac{b^2}{h} \int v ds = \frac{b^2}{2p_\phi} \int (p_\phi d\phi + p_r dr). \quad (92)$$

Now for one complete circuit  $\oint dA = \oint dA' = \pi ab$  and hence

$$\frac{1}{2\pi} \oint (p_\phi d\phi + p_r dr) = \frac{p_\phi}{\sqrt{1 - \epsilon^2}}. \quad (93)$$

We deduce that the eccentricity  $\epsilon$  is independent of time in the adiabatic approximation.

We also have

$$\frac{1}{a(1 - \epsilon^2)} = \frac{\mu S G m_1 m_2}{p_\phi^2}, \quad (94)$$

and we deduce that

$$S(t)a = \text{constant}. \quad (95)$$

In other words the size of the orbit in inertial coordinates  $\mathbf{x} = S(t)\mathbf{r}$  is independent of time.

#### 4.5.2 The effect of the expansion of the universe on the solar system

Expressed in another way, one may use the size of binary systems as a “ruler” with which to measure the “expansion of the universe”. This is consistent with the analysis of the effect of the Slipher-Hubble expansion on the solar system [39].

Since in our Newtonian model the distance to the galaxies  $|\mathbf{x}|$  is increasing in accordance with the Slipher-Hubble law, we see that the solar system has, to the approximation we are working to, a fixed size relative to which the universe may be said to be expanding. One may compare this situation with the well known Einstein-Strauss or Swiss cheese model in General Relativity (see next section for the Newtonian version of this). Each vacuole. i.e. spherical hole in the cheese, is occupied by a locally static Schwarzschild solution. The boundary of the vacuole moves radially outwards with respect to the static Schwarzschild solution with the same motion as a freely moving radial geodesic. Within the vacuole one may imagine a test particle moving on circular geodesic of constant radius. Clearly the boundary of the vacuole is expanding relative to this circular orbit. Thus our Newtonian result, based on the theory of adiabatic invariants, is perfectly consistent with what one obtains according to general relativity.

## 5 Clustering and Swiss Cheese

Most of the work in [2] was concerned with the all when all masses  $m_a$  were taken to be equal. The resulting distributions, for  $N \approx 10^4$  were extremely homogeneous, resembling a random close packing of spheres, with the masses located at the centres of the spheres and with the mean separation mentioned earlier. Interestingly the introduction of a particle with very much larger mass, had the effect of evacuating a much larger sphere, the mass again being located at the centre of the large vacuole, the average density being maintained. This is the Newtonian analogue of Einstein and Strauss’s Swiss Cheese model in General Relativity [40]. It is not what we will later consider as a void.

No evidence was found for clustering or hierarchical structure in the central configurations investigated in [2] and this appears to be consistent with the results of [41] on the absence of clustering in central configurations.

### 5.1 Negative mass and the motion of voids

We shall take (58) as the equation of motion governing the interaction of voids and regions of over density (“attractors”). It has the form of Newton’s law (with respect to the time  $\tau$ ) but for which the effectively Newton’s  $S(\tau)G$  varies with time  $\tau$ . If  $S(t)G \propto \frac{1}{\tau}$  one may, by redefining the time variable, reduce the problem to a time independent Newton’s constant [33, 34, 35]. Unfortunately this is not possible in our case and we have to consider a genuinely time-dependent Newton’s constant. Despite that we can deduce that

- Both voids and attractors fall in the same way in a gravitational field, that is their inertial masses and passive gravitational masses are equal.
- Attractors attract and voids repel. Thus attractors have positive active gravitational mass and voids have negative active gravitational mass. Thus both attractors and voids are attracted towards attractors *and both are repelled from voids*. The direction they actually move of course depends on their initial velocities.
- Attractors have positive inertial masses and positive passive gravitational masses, voids have negative inertial masses and negative passive gravitational masses.
- Action and reaction are equal and opposite and so the centre of mass moves with constant velocity and angular momentum is conserved.

Counter-intuitive motion of this sort appears to have first been contemplated by Föppl [42, 43] before the advent of General Relativity. It is in accordance with the behaviour predicted for general relativity by Bondi [44, 45]. Bondi showed that despite the uniform motion of the barycentre

$$\sum_a m_a \mathbf{x}_a . \quad (96)$$

this could lead to run away solutions. In fact for two bodies with  $m_1 = -m_2$  and  $m = |m_1| = |m_2|$ , (96) is compatible with constant separation

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{d} \quad (97)$$

where  $\mathbf{a}$  is a constant vector. The accelerations of both bodies are given by given by

$$\frac{aGm}{|\mathbf{d}|^2} . \quad (98)$$

In the case considered by Bondi [44, 45] the effective Newton's constant was constant, and hence the mutual acceleration was constant. Bondi succeeded in demonstrating the existence of exact solutions of Einstein's equations exhibiting this effect and it was shown in [46] that negative mass naked singularities could chase regular positive mass black holes (see also [47]). Gravitational repulsion due to uncompensated voids has been pointed out previously by Piran [49]. For other studies of the gravitating properties of negative masses see [50, 48].

## 6 Conclusion

In this paper we have explored the extent to which an analytic treatment of a purely discrete Newtonian particle model can be useful in studying questions in cosmology and large scale structure formation. Our main tool has been what we have referred to as the Dmitriev-Zeldovich equations [6] which are

widely used in numerical simulations. We have given a purely Newtonian point particle derivation. There exist many other approaches based in Newtonian fluid mechanics or a mixture of both fluid and particle viewpoints e.g. [51]. Of course the equations can be obtained as a Newtonian limit of General Relativity and such a treatment may be found in [12, 13, 14].

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